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An Improvement of Hind's Upper Bound on the Total Chromatic Number

AMANDA CHETWYND[1] and ROLAND HÄGGKVIST[2]

[1]Department of Mathematics, University of Lancaster, Lancaster LA1 4YF, UK
[2]Department of Mathematics, University of Umeå, S-901 87 Umeå, Sweden

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We show that the total chromatic number of a simple k-chromatic graph exceeds the chromatic index by at most $18k^{\frac{1}{3}}\log^{\frac{1}{2}} 3k$.

1. Introduction

For notation and additional references see Chetwynd and Häggkvist [2].

Hugh Hind [5, 6] has shown that the total chromatic number for a simple graph with chromatic number k and chromatic index r is at most $r+2\lceil\sqrt{k}\,\rceil$, or in other words, using standard notation, that

Hind's theorem: *For any simple graph G*

$$\chi''(G) \le \chi'(G) + 2\lceil\sqrt{\chi(G)}\,\rceil. \tag{1}$$

The purpose of this note is to improve Hind's theorem for graphs with large chromatic number by essentially reducing the power of χ from $\frac{1}{2}$ to $\frac{1}{3}+\varepsilon$ (the exact statement of our result is given in equations (4)–(6) below).

Our proof uses a lemma which in words states that, if we assign to each vertex x in a k-chromatic r-edge-chromatic multigraph a colour $f(x)$ from $\{1,2,\ldots,r\}$, then there exists a proper $(r+k)$-edge-colouring which on each edge xy assigns a colour from $\{1,2,\ldots,r+k\}-f(x)-f(y)$. Using terminology from our earlier work [2], we can formulate this as

Lemma 1.1. *For any multigraph*

$$\chi''_{1,\chi'} \le \chi' + \chi. \tag{2}$$

Lemma 1.1 is a generalization of the key lemma in Hind [5], and the proof is virtually the same, although the reasoning is slightly more delicate. We defer the proof for a few paragraphs so as not to interrupt the flow of ideas too much, but it shan't be forgotten.

Hugh Hind used his key lemma to bound the total chromatic number from above by the sum of the chromatic index and twice the ceiling of the square root of the maximal degree. We get, in a completely analogous fashion, the stronger statement that for simple graphs

$$\chi''_{1,\Delta} \leq \chi' + 2\lceil\sqrt{\Delta}\,\rceil \tag{3}$$

from which equation (1) follows immediately by the following observation. Given a proper edge-colouring of G using the colours $1,2,\ldots,r$ and a – not even necessarily proper – vertex colouring using the colours $1,2,\ldots,k$, we can partition G into two edge-disjoint graphs: the first uses the edges in colours $k+1, k+2,\ldots,r$; and the other, of maximum degree k and k-edge-chromatic, uses the remaining edges. Using equation (3), we recolour the latter graph using the colours $1,2,\ldots,k$ along with $2\lceil\sqrt{k}\rceil$ new colours without any edge receiving a colour already present on either end. This establishes that (1) follows from (3). (In fact we showed that $\chi''_{1,k} \leq \chi' + 2\lceil\sqrt{k}\,\rceil$ for any k.)

2. Hind's proof

In this note we shall present a slight twist in Hind's proof which makes it possible to improve the bound for simple graphs and deduce that

$$\chi''_{1,\Delta} \leq \chi' + 18\lceil\Delta^{\frac{1}{3}}\log^{\frac{1}{2}} 3\Delta\rceil, \tag{4}$$

from which it follows, in the same way as in the observation above, that

$$\chi''_{1,\chi} \leq \chi' + 18\lceil\chi^{\frac{1}{3}}\log^{\frac{1}{2}} 3\chi\rceil. \tag{5}$$

Thus, in particular, we have, for simple graphs,

$$\chi'' \leq \chi' + 18\lceil\chi^{\frac{1}{3}}\log^{\frac{1}{2}} 3\chi\rceil. \tag{6}$$

The proof of equation (4) follows from Lemma 2.1 below (Lemma 2.1 is a corollary of Lemma 2.3).

Lemma 2.1. *Let G be an m-regular simple graph and k an integer such that $2^{3k-1} \leq m < 2^{3k+2}$. Assume furthermore that $\frac{m}{60\log 3m} \geq 2^k$. Then there exists a partition of the vertices in G into $2^k < 2\lceil m^{\frac{1}{3}}\rceil$ parts such that the degree within each part differs from $m/2^k$ by at most $4\lceil m^{\frac{1}{3}}\log^{\frac{1}{2}} 3m\rceil$.*

Proof of (4) using Lemma 2.1 and Lemma 1.1. We may assume that G is regular of degree m since every r-edge-chromatic graph with maximum degree m can be embedded in an m-regular r-edge-chromatic graph. Moreover, we may assume that $18\lceil m^{\frac{1}{3}}\log^{\frac{1}{2}} 3m\rceil \leq 2\lceil\sqrt{m}\rceil$, since otherwise equation (3) applies. Thus, after some easy calculations and with k chosen such that $2^{3k-1} \leq m < 2^{3k+2}$,

$$\frac{m}{2^k} \geq \frac{m^{\frac{2}{3}}}{2} \geq m^{\frac{1}{3}} \geq 81\log 3m$$

so that the assumptions in Lemma 2.1 are fulfilled for m and k. Let the vertices of G be assigned (forbidden) colours from $1,\ldots,m$; the forbidden colour on x is $f(x)$.

Assume now that G is given a vertex partition as in Lemma 2.1 with parts $V_1, V_2, \dots, V_{2^k}$. Consider the graph H obtained from G by deletion of all edges within each part. Note that H has maximum degree at most $m - t$ with $t = \lfloor m/2^k \rfloor - 4\lceil m^{\frac{1}{3}} \log^{\frac{1}{2}} 3m \rceil$, and that H has chromatic number at most 2^k. Use of equation (2) and Vizing's bound $\chi' \leq \Delta + 1$ [9] shows that H admits a proper edge-colouring Λ_1 using as colours the symbols in $\{1, 2, \dots, m - t + 1\}$ together with 2^k new colours $\{\alpha_1, \alpha_2, \dots, \alpha_{2^k}\}$ in such a way that no vertex x is incident with any edge coloured $f(x)$. Let G_i denote the graph $G[V_i]$. It follows directly from Lemma 2.1 that G_i has maximum degree at most $\lceil m/2^k \rceil + 4\lceil m^{\frac{1}{3}} \log^{\frac{1}{2}} 3m \rceil \leq 1 + t + 8\lceil m^{\frac{1}{3}} \log^{\frac{1}{2}} 3m \rceil$ and (by Vizing's Theorem again), it has chromatic index at most $\Delta(G_i)+1$. Therefore, by equation (3), each G_i admits a proper edge-colouring Λ_2 using the colours in $m-t+2, m-t+3, \dots, m+8\lceil m^{\frac{1}{3}} \log^{\frac{1}{2}} 3m \rceil + 3$ (the colours $m+1, m+2, \dots, m + 8\lceil m^{\frac{1}{3}} \log^{\frac{1}{2}} 3m \rceil + 3$ are new colours), together with some additional new colours $\{\alpha_{2^k+1}, \alpha_{2^k+2}, \dots, \alpha_{2^k+2s}\}$ where $s = 2^{k+1} + \lceil 2 \log^{\frac{1}{2}} 3m \rceil \leq 4\lceil m^{\frac{1}{3}} \log^{\frac{1}{2}} 3m \rceil$ (note that $s^2 > 1 + t + 8\lceil m^{\frac{1}{3}} \log^{\frac{1}{2}} 3m \rceil \geq \Delta(G_i)$) in such a way that no vertex x is incident with any edge of colour $f(x)$. This proved equation (4), since we have used in total less than $m + 8\lceil m^{\frac{1}{3}} \log^{\frac{1}{2}} 3m \rceil + 3 + 2^k + 8\lceil m^{\frac{1}{3}} \log^{\frac{1}{2}} 3m \rceil \leq m + 18\lceil m^{\frac{1}{3}} \log^{\frac{1}{2}} 3m \rceil$ colours. □

We now turn to the proofs of Lemmas 1.1 and 2.1.

Proof of Lemma 1.1. It clearly suffices to show that if S is an independent set of vertices in a r-edge-coloured multigraph G and we have assigned to each vertex s in S a colour $f(s)$ from the colours used in the edge colouring, then the edges of G can be recoloured using the old colours as well as one new colour α such that

(a) at each vertex s in S the colour $f(s)$ fails to appear on any edge incident with s,

(b) every vertex not in S is incident with the same colours as in the old colouring except that one colour can have been changed to α, and

(c) every edge with colour α has one end in S.

We may as well assume that the original edge colouring, Λ, *say* used integers $1, 2, \dots r$ as colours.

We find the wanted recolouring of Λ inductively. There is nothing to prove when S is the empty set, so we can assume that we have found such a recolouring Λ_{t-1} satisfying (a), (b) and (c) for the independent set S_{t-1} consisting of the vertices s of S, where $f(s) < t$ and, moreover, s is incident with an edge coloured $f(s)$. Thus, in particular, no vertex in $S \setminus S_{t-1}$ is incident with any edge coloured α. Let S_t be the union of S_{t-1} and those vertices s in S for which $f(s) = t$, and which, moreover, are incident with an edge coloured t. We shall show how to obtain an edge colouring Λ_t satisfying (a), (b) and (c) for S_t. To do so we let B_t consist of the subgraph of G consisting of the edges of colours α and t in the edge colouring Λ_{t-1}. It is important to note that every component in B_t incident with $S_t \setminus S_{t-1}$ is an alternating path which begins with a t-edge at $S_t \setminus S_{t-1}$, that every such component has internal vertices in S_{t-1} or else ends with an α-edge incident with S_{t-1}, and finally that each such component which ends with a t-edge does so at a vertex not incident with S. Recolouring the components in B_t incident with $S_t \setminus S_{t-1}$ by swapping α and t-edges gives us the desired edge colouring Λ_t. It is important to note that no vertex in S_{t-1} gets a new conflict, since t is allowed at any vertex in S_{t-1}. It should also be noted that if a vertex outside S is non-incident with t in Λ_{t-1}, then it is also non-incident with t in Λ_t. Thus (a), (b) (and clearly (c)) hold for Λ_t. Eventually, then, Λ_r is constructed and the process stops. □

Proof of Lemma 2.1. We prove two lemmas, the second of which contains Lemma 2.1 as a corollary.

Lemma 2.2. *Let G be a simple graph with maximum degree m. Then there exists a partition of the n vertices in G into 2 parts such that the degree within each part differs from its expected value by at most* $\frac{1}{2}\sqrt{m \log 3m}$.

In the proof of Lemma 2.2, we shall use the following version of the Erdös–Lovász local lemma [3] (can also be found as exercise 2.18 in Lovász [7] and in Spencer [6]) and some standard bounds on the tail of the binomial distribution.

The Erdös–Lovász local lemma. *Let* $A_1, A_2, \ldots, A_n$ *be events in a probability space. Let* Γ *be a graph on* $\{1, 2, \ldots, n\}$ *such that* $ij \in E(\Gamma)$ *whenever* A_i *depends on* A_j *(i.e. each* A_i *is mutually independent of all except at most* $\Delta(\Gamma)$ *other events* A_j*). Assume that* $Pr(\bar{A}_i) < \frac{1}{4\Delta(\Gamma)}$. *Then* $Pr(A_1 A_2 \cdots A_n) > 0$.

Proof of Lemma 2.2. Consider all possible bipartitions of $V(G)$; there are obviously exactly 2^n such bipartitions. Let the vertices of G be $v_1, v_2, \ldots, v_n$ and let A_i be the event that the neighbours of v_i are split roughly evenly so that the discrepancy is less than $\alpha = \frac{1}{2}\sqrt{m \log 3m}$ between the two parts in the bipartition. Now note that A_i is independent of all the events A_j, where the corresponding pair of vertices in G are of distance more than 2 apart. Thus the dependency graph Γ has maximum degree at most m^2. We want to establish that there exists a bipartition among our family of bipartitions such that all the events A_i have happened. By the Erdös–Lovász local lemma, it suffices to verify that $Pr(\bar{A}_i) < 1/4\Delta(\Gamma)$ or in other words that $Pr(A_i) \geq 1 - 1/4m^2$. We finish the proof of the lemma by proving this fact.

Consider a fixed i and let $B = N(v_i)$. Our bipartitions define n independent random variables $X_1, X_2, \ldots, X_n$, each labelled by a vertex in G, by the following rule. Let the colours used in each bipartition be red and blue. For a fixed bipartition we let X_j be 0 if $v_j \notin B$ and if $v_j \in B$ then X_j is $+1$ or -1, depending on whether or not v_j is in the red or blue part. Let $X = \sum X_i$ and $X^* = \sum_{X_i \neq 0} X_i$. Note that

$$Pr(A_i) = Pr(|X| < 2\alpha) = Pr(|X^*| < 2\alpha).$$

Now, X^* is the sum of at most m independent random variables, each taking $+1$ or -1 with probability $1/2$, and therefore by the Chernoff estimates [1] (also in Häggkvist and Thomason [4, Lemma 2], or Spencer [8, p. 29])

$$Pr(|X^*| < \lambda) \geq 1 - 2e^{-2\lambda^2/m}.$$

In particular, for $\lambda = 2\alpha$ we get

$$Pr(A_i) \geq 1 - 2e^{-8\alpha^2/m} \geq 1 - \frac{1}{4m^2}$$

by our choice of α. $\square$

Lemma 2.3. *Let G be an m-regular simple graph. Then, for every integer $p = 2^k$ such that $\frac{m}{60 \log 3m} \geq 2^k$ there exists a partition of the n vertices in G into p parts such that the degree within each part differs from its expected value m/p by at most $3\sqrt{\frac{m}{p}\log 3m}$.*

Proof. By induction on k and using Lemma 2.2. The case $k = 0$ is obviously true and the case $k = 1$ is immediate from Lemma 2.2, since $\frac{1}{2}\sqrt{m \log 3m}$ is less than $3\sqrt{\frac{m}{2}\log 3m}$. Assuming that the lemma has been established for $p = 2^{k-1}$ we prove it for $p = 2^k$. The truth of the lemma when $p = 2^{k-1}$ implies that G admits a partition into 2^{k-1} parts such that the degree within each part differs from $m/2^{k-1}$ by at most $3\sqrt{\frac{m}{2^{k-1}}\log 3m}$. Let G_i be the subgraph induced by the vertices in the ith part and assume that G_i has n_i vertices. By Lemma 2.2 we can divide each G_i into two parts such that each vertex v has degree within each part differing from $\frac{1}{2}d(v, G_i)$ by at most $\frac{1}{2}\sqrt{\Delta(G_i)\log 3\Delta(G_i)}$. We first of all note that $\frac{1}{2}d(v, G_i)$ differs from $m/2^k$ by at most

$$\frac{3}{2}\sqrt{\frac{m}{2^{k-1}}\log 3m} = \frac{3}{\sqrt{2}}\sqrt{\frac{m}{2^k}\log 3m}.$$

We also note that, because $\sqrt{a^2 + b} \leq a + \frac{b}{2a}$, $a, b \geq 0$, we have

$$\begin{aligned} \frac{1}{2}\sqrt{\Delta(G_i)\log 3\Delta(G_i)} &\leq \frac{1}{2}\sqrt{\left(\frac{m}{2^{k-1}} + 3\sqrt{\frac{m}{2^{k-1}}\log 3m}\right)\log 3m} \\ &\leq \frac{1}{2}\sqrt{\log 3m}\left(\sqrt{\frac{m}{2^{k-1}}} + \frac{3}{2}\sqrt{\log 3m}\right) \\ &= \frac{\sqrt{2}}{2}\sqrt{\frac{m}{2^k}\log 3m} + \frac{3}{4}\log 3m. \end{aligned}$$

Since $\frac{3}{\sqrt{2}} + \frac{\sqrt{2}}{2} < 2.9$ we obtain the final condition of Lemma 2.3 provided that, for example, $\frac{3}{4}\log 3m \leq 0.1\sqrt{\frac{m}{2^k}\log 3m}$. This is true when

$$\frac{m}{2^k} \geq \frac{900}{16}\log 3m,$$

a condition met by the assumptions of Lemma 2.3. $\square$

Finally, we are ready to prove Lemma 2.1. It suffices to show that, under the assumptions in Lemma 2.1,

$$4\lceil m^{\frac{1}{3}}\log^{\frac{2}{3}} 3m\rceil \geq 3\sqrt{\frac{m}{2^k}\log 3m},$$

an inequality which holds since after raising both sides to the sixth power and clearing powers of $\log 3m$ we have $4^6 m^2 > 3^6 m^3/2^{3k}$ and this is a consequence of $(4/3)^6 2^{3k} > 2^{3k+3} > m$. □

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